[11:00-11:50] Sinusoidal response of FIR filters (slides 4-6)

The impulse response $h[n]$ uniquely and fully characterizes an LTI system (if the system is LTI, then we know that the initial conditions must be zero).

The frequency response $H(e^{j\hat{\omega}})$ is directly related to the impulse response and tells us how a complex sinusoidal input is modified by the system.

$$x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$$

$$y[n] = Ae^{j\phi}e^{j\hat{\omega}n}H(e^{j\hat{\omega}}) = A\underbrace{|H(e^{j\hat{\omega}})|}_{\substack{\text{magnitude}\\ \text{response}}}e^{j(\phi+\overbrace{\angle H(e^{j\hat{\omega}})}^{\substack{\text{phase}\\ \text{response}}})}e^{j\hat{\omega}n}$$

Example: $h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$

$$\begin{aligned} H(e^{j\hat{\omega}}) &= \sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k} \\ &= 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} \\ &= e^{-j\hat{\omega}}(e^{j\hat{\omega}} + 2 + e^{-j\hat{\omega}}) \\ &= \underbrace{(2+2\cos(\hat{\omega}))}_{\text{magnitude response}} \quad \overbrace{e^{-j\hat{\omega}}}^{\text{phase response is } -\hat{\omega}} \end{aligned}$$

[11:50-12:00] Sinusoidal response of FIR filters (slides 6-7)

An LTI system cannot create frequencies that are not in the input:

$$x[n] = e^{j\hat{\omega}n} \rightarrow y[n] = H(e^{j\hat{\omega}})e^{j\hat{\omega}n}$$

For real-values impulse response $h[n]$, the frequency response is conjugate symmetric:

$$H(e^{j\hat{\omega}}) = \sum_{n=0}^{M} h[n]\, e^{-j\hat{\omega}n} = \sum_{n=0}^{M} h[n]\, (e^{j\hat{\omega}})^{-n}$$

$$H^*(e^{j\hat{\omega}}) = \left(\sum_{n=0}^{M} h[n]\, e^{-j\hat{\omega}n}\right)^* = \sum_{n=0}^{M} h^*[n]\, (e^{-j\hat{\omega}n})^* = \sum_{n=0}^{M} h[n]\, e^{j\hat{\omega}n} = H(e^{-j\hat{\omega}})$$

Because $h[n]$ is real-valued, the complex conjugate of $h[n]$ is $h[n]$.

Since $2\cos(\hat{\omega}n) = e^{-j\hat{\omega}n} + e^{j\hat{\omega}n}$, we can also use the frequency response to find the output corresponding to a real-values cosine input.

$$x[n] = \frac{1}{2}\left(e^{-j\hat{\omega}n} + e^{j\hat{\omega}n}\right) \to y[n] = \frac{1}{2}H^*\left(e^{j\hat{\omega}}\right)e^{-j\hat{\omega}n} + \frac{1}{2}H\left(e^{j\hat{\omega}}\right)e^{j\hat{\omega}n}$$

$$x[n] = \cos(\hat{\omega}n) \to y[n] = \left|H\left(e^{j\hat{\omega}}\right)\right| \cos\left(\hat{\omega}n + \angle H\left(e^{j\hat{\omega}}\right)\right)$$

[12:00] Ideal delay $y[n] = x[n-1]$ for $n \geq 0$. Initial condition of $x[-1]$ must be 0 as a necessary condition for linear and time-invariant properties to hold.

Impulse response: $h[n] = \delta[n-1]$	Frequency response: $H\left(e^{j\hat{\omega}}\right) = e^{-j\hat{\omega}}$
Magnitude response $\left|H\left(e^{j\hat{\omega}}\right)\right| = 1$ (allpass)	Phase response $\angle H\left(e^{j\hat{\omega}}\right) = -\hat{\omega}$ (linear phase)

[12:05] Two-point averaging filter $y[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$

$h[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]$	
$H\left(e^{j\hat{\omega}}\right) = \frac{1}{2} + \frac{1}{2}e^{-j\hat{\omega}} = \frac{1}{2}e^{-j\frac{\hat{\omega}}{2}}\left(e^{j\frac{\hat{\omega}}{2}} + e^{-j\frac{\hat{\omega}}{2}}\right) = \cos\left(\frac{\hat{\omega}}{2}\right)e^{-j\frac{\hat{\omega}}{2}}$	
$\left|H\left(e^{j\hat{\omega}}\right)\right| = \cos\left(\frac{\hat{\omega}}{2}\right)$	$\angle H\left(e^{j\hat{\omega}}\right) = -\frac{\hat{\omega}}{2}$

[12:10] First-order difference $y[n] = \frac{1}{2}x[n] - \frac{1}{2}x[n-1]$

$h[n] = \frac{1}{2}\delta[n] - \frac{1}{2}\delta[n-1]$	
$H\left(e^{j\hat{\omega}}\right) = \frac{1}{2} - \frac{1}{2}e^{-j\hat{\omega}} = \frac{1}{2}e^{-j\frac{\hat{\omega}}{2}}\left(e^{j\frac{\hat{\omega}}{2}} - e^{-j\frac{\hat{\omega}}{2}}\right) = j\sin\left(\frac{\hat{\omega}}{2}\right)e^{-j\frac{\hat{\omega}}{2}} = \sin\left(\frac{\hat{\omega}}{2}\right)e^{j\left(\frac{\pi}{2}-\frac{\hat{\omega}}{2}\right)}$	
$\left|H\left(e^{j\hat{\omega}}\right)\right| = \begin{cases} \sin\left(\frac{\hat{\omega}}{2}\right) & \text{for } 0 \leq \hat{\omega} < \pi \\ -\sin\left(\frac{\hat{\omega}}{2}\right) & \text{for } -\pi \leq \hat{\omega} < 0 \end{cases}$	$\angle H\left(e^{j\hat{\omega}}\right) = \begin{cases} \frac{\pi}{2} - \frac{\hat{\omega}}{2} & \text{for } 0 \leq \hat{\omega} < \pi \\ -\frac{\pi}{2} - \frac{\hat{\omega}}{2} & \text{for } -\pi \leq \hat{\omega} < 0 \end{cases}$